Advanced ODE-Lecture 6
General Theory of Linear Systems

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Motivation

- An important class of systems with global existence;
- A good approximation to nonlinear system near a known solution by linearization;
- Algebraic structure for the whole solutions set of homogeneous linear systems;
- Applications to nonlinear systems;
- Linear system theory is almost complete except for a few remained open.

Global Existence

Consider the IVP:

$$\dot{x} = A(t)x + h(t), \quad x(t_0) = x_0, \quad x \in R^n,$$

where $A(t)$ and $h(t) \in C(I)$, $I = (a, b)$ ($a = -\infty$ or $b = \infty$ is permitted).

Theorem 6.1 ((Fundamental Theorem) Suppose that $A(t)$ and $h(t) \in C(I)$, where $I = (a, b)$. If $t_0 \in I$, then, the IVP of linear systems has global solutions on (a, b) for all $x_0 \in R^n$.

Proof. Since $f(t, x) = A(t)x + h(t) \in C(I \times R^n)$ is locally Lipschitz, then, there exists a unique solution $x(t, t_0, x_0)$ of the IVP defined on $I_{\max}$. If $I_{\max} \neq (a, b)$, say $I_{\max}^{+} = [t_0, \omega_+)$ with $\omega_+ < b$, we show by contradiction.

Let $t \in [t_0, \omega_+)$. Then one has

$$x(t,t_0,x_0) - x_0 = \int_{t_0}^{t} \{A(s)x(s,t_0,x_0) + h(s)\}\, ds .$$

$$\Rightarrow \quad \| x(t,t_0,x_0) \| \le \| x_0 \| + \int_{t_0}^{t} \| A(s) \| \| x(s,t_0,x_0) \| \, ds + \int_{t_0}^{\omega_+} \| h(s) \| \, ds$$

Application of Gronwall inequality yields

$$\| x(t,t_0,x_0) \| \le \{ \| x_0 \| + \int_{t_0}^{\omega_+} \| h(s) \| \, ds \} \cdot \exp(\int_{t_0}^{t} \| A(s) \| \, ds)$$

$$\le \{ \| x_0 \| + \int_{t_0}^{\omega_+} \| h(s) \| \, ds \} \cdot \exp(\int_{t_0}^{\omega_+} \| A(s) \| \, ds) < \infty .$$

Thus, $\| x(t,t_0,x_0) \|$ is bounded on $I_{\max}^{+}$. However, we have $\| x(t,t_0,x_0) \| \to \infty$ as $t \to \omega_+^-$ by the extensibility theorem. This is a contradiction. It is similar to show the left case of $I_{\max}^{-} = (\omega_-, t_0]$ if $\omega_- > a$. □

Remark 6.1 Linear system has global solutions for whole solutions, which can be regarded as a set because of global existence for all.

Superposition Principle

Theorem 6.2 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h_1(t)$ and $x' = A(t)x + h_2(t)$, respectively. Then, $c_1x_1(t) + c_2x_2(t)$ is a solution of

$$x' = A(t)x + (c_1h_1(t) + c_2h_2(t)) .$$

Remark 6.2 From math point of view, Superposition Principle is quite simple. However, it is a main characterization of linear systems. Extremely important in practice! From control point of view, if we regard $h(t)$ as an input and $x(t)$ response (output), then, any dynamic system satisfying superposition principle is linear. This principle can be verified based on experiment without a form of equations.

Corollary 6.1 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x$. Then $c_1x_1(t) + c_2x_2(t)$ is a solution of $x' = A(t)x$.

Corollary 6.2 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h(t)$ and $x' = A(t)x$ respectively. Then $x_1(t) + x_2(t)$ is a solution of $x' = A(t)x + h(t)$.

Corollary 6.3 Suppose that $x_1(t)$ and $x_2(t)$ are solutions of $x' = A(t)x + h(t)$. Then $x_1(t) - x_2(t)$ is a solution of $x' = A(t)x$.

General Theory of Linear Systems

$$x' = A(t)x\,.$$

Let Ω be the set of whole solutions.

Theorem 6.3 Ω is an n-dimensional vector linear space.

Proof. For any $t_0 \in I$ and $x_0 \in R^n$, there exists a unique solution $x(t, x_0)$, $t \in I$.

Define a map $T: R^n \to \Omega$ as follows.

$$T(x_0) = x(t, x_0)\,.$$

We first show that T is a linear map. We know by Superposition Principle that

$$c_1 x(t, x_0^1) + c_2 x(t, x_0^2) \in \Omega \quad \text{if} \quad x(t, x_0^1), x(t, x_0^2) \in \Omega$$

and it satisfies the initial value $c_1 x_0^1 + c_2 x_0^2$. By uniqueness,

$$x(t, c_1 x_0^1 + c_2 x_0^2) = c_1 x(t, x_0^1) + c_2 x(t, x_0^2)\,.$$

Then,

$$T(c_1x_0^1 + c_2x_0^2) = x(t, c_1x_0^1 + c_2x_0^2) = c_1x(t, x_0^1) + c_2x(t, x_0^2) = c_1T(x_0^1) + c_2T(x_0^2).$$

Therefore T is a linear map.

We then show that T is an isomorphic map.

For any $x(t) \in \Omega$, there exists $x_0 \in R^n$ such that $T(x_0) = x(t)$. Therefore, T is onto.

For any $x(t, x_0^1)$, $x(t, x_0^2) \in \Omega$ with $x(t, x_0^1) \neq x(t, x_0^2)$, it must be $x_0^1 \neq x_0^2$. It is also true inversely.

Therefore, T is one-to-one. Combination of these two results in T is isomorphic.

$\Rightarrow \ \Omega \cong R^n$. $\square$

Remark 6.3 Geometric explanation of $T: R^n \to \Omega$: any integral curve $x(t) \in \Omega$ always intersects uniquely the super-plane $t = t_0$ at some point $x_0 \in R^n$.

Remark 6.4 Since $\Omega \cong R^n$, the algebraic structure of Ω is clear. Any n linearly independent elements (solutions, vector functions) of Ω form a base of Ω.

Definition 6.1 $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ ($t \in I$) is said to be **linearly independent** in I if $\sum_{j=1}^{n} c_j x_j(t) \equiv 0$ for all $t \in I$ implies that $c_1 = c_2 = \cdots c_n = 0$. Conversely, these vector-valued functions are said to be **linearly dependent** in I if there exist $c_1, c_2, \cdots, c_n$ not all zero s.t. $\sum_{j=1}^{n} c_j x_j(t) \equiv 0$ for all $t \in I$.

Definition 6.2 Any solutions $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ ($t \in I$) that are linearly independent is said to be **a fundamental set of solutions**.

Example 6.1 Show that $n+1$ vector-valued functions

$$x_1(t)=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\in R^n,\ x_2(t)=\begin{pmatrix}t\\0\\\vdots\\0\end{pmatrix}\in R^n,\cdots,\text{ and }\ x_{n+1}(t)=\begin{pmatrix}t^n\\0\\\vdots\\0\end{pmatrix}\in R^n$$

are linearly independent on any interval I.

Proof by contradiction. If there exist $c_1, c_2, \cdots, c_{n+1}$ not all zero s.t.

$$\sum_{j=1}^{n+1} c_j x_j(t) \equiv 0 \quad \text{for all} \quad t \in I,$$

then we have $c_1 + c_2 t + \cdots + c_{n+1} t^n \equiv 0$ for all $t \in I$. But there are at most n roots only for this polynomial equation according to the fundamental theorem of algebra unless $c_1 = c_2 = \cdots = c_{n+1} = 0$. This is a contradiction.

Remark 6.5 This example seems contradiction to Theorem 6.3, which states that there only exist n linearly independent vector-valued solutions. In fact, it is not. Why?

Theorem 6.4 (General Solution Algebraic Structure) Let $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ be a fundamental set of solutions. Then the general solution is

$$x(t) = \sum_{j=1}^{n} c_j x_j(t),$$

where $c_1, c_2, \cdots, c_n$ are any arbitrary constants. Moreover, it includes the whole solutions.

Proof. Take a base of R^n as follows.

$$x_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad x_n(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The corresponding solutions are as follows.

$$x_1(t), x_2(t), \cdots, x_n(t) \in \Omega, \quad t \in I.$$

It remains to show that $x_1(t), x_2(t), \cdots, x_n(t)$ is a fundamental set of solutions. If

$$\sum_{j=1}^{n} c_j x_j(t) \equiv 0, \quad t \in I,$$

there exists a map $T: R^n \to \Omega$ such that $T\{x_j(t_0)\} = x_j(t)$ $(j = 1, 2, \quad, n)$ by Theorem 6.3. Then

$$\sum_{j=1}^{n} c_j T\{x_j(t_0)\} = 0.$$

Since T is linear, we have $T\{\sum_{j=1}^{n} c_j x_j(t_0)\} = 0$. Then we have $\sum_{j=1}^{n} c_j x_j(t_0) = 0$. $\Rightarrow$

$c_1 = c_2 = \quad = c_n = 0$. Therefore, $x_1(t), x_2(t), \cdots, x_n(t)$ are linearly independent in I.

We conclude that $x(t) = \sum_{j=1}^{n} c_j x_j(t)$ is a general solution.

Moreover, it includes the whole solution because for $\forall\, x(t) \in \Omega$ with $x(t_0)$, there exist c_j^0, $j = 1, 2, \cdots, n$ s.t. $x(t_0) = \sum_{j=1}^{n} c_j^0 x_j(t_0)$ since $\{x_j(t_0)\}$ is a basis of R^n.

Then,

$$x(t) \equiv \sum_{j=1}^{n} c_j^0 x_j(t) \text{ by uniqueness.}$$

This shows that the general solution includes the whole solution. □

Definition 6.3 Let $\Phi(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. $\Phi(t)$ is said to be **a matrix solution** if $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$. $\Phi(t)$ is said to be **a fundamental matrix solution** if $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ are linearly independent. A matrix solution $\Phi(t)$ is said to be **a principle matrix solution** if $\Phi(t_0) = I_n$, which is denoted as $\Phi(t, t_0)$.

Theorem 6.5 (Matrix Form of General Solution) The general solution of $x' = A(t)x$ is

$$x(t) = \Phi(t)c,$$

where $\Phi(t)$ is a fundamental matrix solution and $c = (c_1, c_2, \cdots, c_n)^T$ is an arbitrary n vector. The solution of the IVP $x' = A(t)x$ with $x(t_0) = x_0$ is

$$x(t) = \Phi(t, t_0)x_0 .$$

Lemma 6.1 A matrix solution $\Phi(t)$ is a fundamental matrix solution $\Leftrightarrow$ $\det \Phi(t) \neq 0$ for all $t \in I$.

Proof. It is noted that $x_1(t), x_2(t), \cdots, x_n(t) \in \Omega$ are linearly independent $\Leftrightarrow$

$$\det(x_1(t), x_2(t), \cdots, x_n(t)) \neq 0 \text{ for all } t \in I. \quad \square$$

Definition 6.4 $W(t) = \det \Phi(t)$ is said to be **a Wroskian Determinant**.

Remark 6.6 $\Phi(t)=\begin{pmatrix}1 & t & t^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $t\in(-\infty,\infty)$; but $\det\Phi(t)\equiv 0$, $t\in(-\infty,\infty)$. What is implied from this example?

Theorem 6.6 (Liouville Formula) Suppose that $\Phi(t)$ is a matrix solution of $x'=A(t)x$. Then

$$\det\Phi(t)=\det\Phi(t_0)\exp\{\int_{t_0}^{t} trA(s)ds\},$$

where $trA(t)=\sum_{j=1}^{n}a_{jj}(t)$ is a trace of $A(t)$, $t\in I$. (**Homework-1**)

Remark 6.7 Liouville Formula implies that if $\Phi(t)$ is a matrix solution of $x'=A(t)x$, then

$$\det\Phi(t)\neq 0 \text{ for all } t\in I \iff \det\Phi(t_0)\neq 0 \text{ for some } t_0\in I;$$

$$\det\Phi(t)\equiv 0 \text{ for all } t\in I \iff \det\Phi(t_0)=0 \text{ for some } t_0\in I.$$

Theorem 6.7 A matrix solution $\Phi(t)$ is a fundamental matrix solution $\Leftrightarrow$ there exists a point $t_0 \in I$ s.t. $\det \Phi(t_0) \neq 0$.

Theorem 6.8 (Properties of $\Phi(t, t_0)$) (Homework-2)

1) $\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$;

2) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$;

3) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$;

4) $x(t, t_0, x_0) = \Phi(t, t_0)x_0$

2. Non-Homogeneous Linear Systems

$$x' = A(t)x + h(t), \quad x(t_0) = x_0.$$

Theorem 6.9 (General Solution Structure for the Non-Homogeneous) Suppose $x^*(t)$ is a particular solution of $x' = A(t)x + h(t)$; $\Phi(t)$ is a fundamental matrix solution of its corresponding homogeneous linear system $x' = A(t)x$. Then the general solution of $\dot{x} = A(t)x + h(t)$ is given by

$$x(t) = \Phi(t)c + x^*(t),$$

where c is an arbitrary vector constant. Moreover it includes the whole solutions.

Proof. By the superposition principle (Corollary 6.2), $\Phi(t)c + x^*(t)$ is a solution of $\dot{x} = A(t)x + h(t)$. Since $\dfrac{\partial x(t)}{\partial c} = \Phi(t)$ is nonsingular for all $t \in I$, $\Phi(t)c + x^*(t)$ is a general solution too.

Next, we show that $\Phi(t)c + x^*(t)$ includes the whole solution of $x' = A(t)x + h(t)$.

For any solution $\bar{x}(t)$ of $x' = A(t)x + h(t)$, Take $c_0 = \Phi^{-1}(t_0)(\bar{x}(t_0) - x^*(t_0))$, it follows that $\bar{x}(t)$ and $x(t) = \Phi(t)\Phi^{-1}(t_0)(\bar{x}(t_0) - x^*(t_0)) + x^*(t)$ have the same initial value condition $x(t_0) = \bar{x}(t_0)$ and so we find c_0 s.t.

$$x(t) = \Phi(t)c_0 + x^*(t) \equiv \bar{x}(t).$$

This shows that $\bar{x}(t)$ is an element of the general solutions. ■

Remark 6.8 $x^*(t)$ can be determined by $\Phi(t)$ by the method of **Variation of Constants.**

Theorem 6.10 (Variation of Constants) The general solution of $\dot{x} = A(t)x + h(t)$ is given by

$$x(t) = \Phi(t)c + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s\,;$$

The IVP of $\dot{x} = A(t)x + h(t)$ with $x(t_0) = x_0$ is given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s$$

$$= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)h(s)\,\mathrm{d}s\,,$$

where $\Phi(t)$ is a fundamental matrix solution and $\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$ is a principle matrix solution.

Proof. Suppose that $x(t)=\Phi(t)c(t)$ is a solution of $\dot{x}=A(t)x+h(t)$, where $c(t)$ will be determined. Substituting $x(t)=\Phi(t)c(t)$ into $\dot{x}=A(t)x+h(t)$, we have

$$\Phi(t)c'(t)=h(t).$$

Then,

$$c(t)=\int_{t_0}^{t}\Phi^{-1}(s)h(s)\,\mathrm{d}s.$$

From this, it follows that

$$x^*(t)=\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)h(s)\,\mathrm{d}s.$$

By the general solution structure, it yields

$$x(t)=\Phi(t)c+\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)h(s)\,\mathrm{d}s,$$

where c is an arbitrary vector constant. If $x(t_0)=x_0$ is satisfied, $c=\Phi^{-1}(t_0)x_0$ is determined. Then

$$x(t)=\Phi(t)\Phi^{-1}(t_0)x_0+\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)h(s)\,\mathrm{d}s$$

$$=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,s)h(s)\,\mathrm{d}s. \quad \square$$

Nonlinear Systems with Nonhomogeneous Linear Structure

Consider the nonlinear system as follows.

$$\dot{x} = A(t)x + f(t,x), \quad x(t_0) = x_0 .$$

Its equivalent integral form is given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)f(s,x(s))\,\mathrm{d}s .$$

The proof is left for homework. This structure of integral form is similar to Variation of Constants Formula. However its proof is not the same.

This result is useful in future of study for nonlinear systems.

Computation of $\Phi(t)$

By Theorem 6.10, the computation of $\Phi(t)$ is key for the computation of the IVP of the linear systems (L-1). Is there any way to compute $\Phi(t)$? The answer is negative in general. See the example of Ricatti equation

$$x' = t^2 + x^2,$$

which has no explicit solution. Taking a transformation: $x = -\frac{u'}{u}$, we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We conclude that we are not able to solve this particular linear time-varying system. Otherwise, we can solve the Ricatti equation. There is no way to find a systematic method to solve $\Phi(t)$! If we restrict ourselves by $A(t) \equiv A$, we are able to find its fundamental matrix solution as $\Phi(t) = e^{At}$. There exist many ways to solve $\Phi(t) = e^{At}$! These are the contents of Elementary ODE.

Summary

- **Linear system has global solutions;**
- **Linear system has superposition principle, which is a key characterization of linear dynamics;**
- **Linear system has an important algebra property: $\Omega \cong R^n$, finite dimension;**
- **Linear system has general solution algebraic structure formulae.**
- **How to find a fundamental matrix solution $\Phi(t)$ in general remains unsolved.**

Homework

1. Show that $\dot{x} = A(t)x + h(t)$ has only $n+1$ linearly independent solutions, where $h(t)$ is not identically zero on I; $A(t)$ and $h(t)$ are continuous on I.

2. Show that the IVP

$$x' = A(t)x + f(t,x), \quad x(t_0) = x_0$$

and the integral equations

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) f(s, x(s))\,\mathrm{d}s$$

are equivalent. That is, they have the same set of solutions, where $\Phi(t)$ is a fundamental matrix solution, $A(t)$ is continuous on I and $f(t,x)$ is continuous on $I \times R^n$.

3. Do Homework 1-2.